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On AB Bond Percolation on the Square Lattice and AB Site Percolation on Its Line Graph

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We prove that AB site percolation occurs on the line graph of the square lattice when $p \in (1-\sqrt{1-p_c}, \sqrt{1-p_c})$, where p_c is the critical probability for site percolation in $\mathbb{Z}^2$. Also, we prove that AB bond percolation does not occur on $\mathbb{Z}^2$ for $p = \frac{1}{2}$.

KEY WORDS: Half close-packed graph of $\mathbb{Z}^2$; AB percolation; stochastic domination; randomly oriented lattice.

1. INTRODUCTION AND RESULTS

The AB percolation model was first studied in refs. 1 and 2 (also, earlier paper[(3)] contains some simulation results for the AB percolation model on the triangular lattice). The general setting for this model is the following. Let $G = (V_G, E_G)$ be the graph with vertex set V_G and edge set E_G. Into any vertex $v \in V_G$, independently, we put a label "A" with probability p, and put a label "B" with probability $1-p$; $X(v) \in \{A, B\}$ stands for the state of v. We call a self-avoiding path $\pi = \{v_0, e_0, v_1, e_1,..., e_{n-1}, v_n\}$ (where e_i is the edge between v_i and v_{i+1}) on G an AB path if $X(v_i) \neq X(v_{i+1})$ for all $0 \leqslant i < n$; and we say that AB percolation occurs if there exists an infinite AB path on G.

The first question that one may ask is: On which graph and when does AB percolation occur? Let us mention some known results concerning that. Halley[(1)] proved that if the graph G is bipartite and has critical probability

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for site percolation strictly greater than $\frac{1}{2}$, then AB percolation does not occur for $p=\frac{1}{2}$. Appel and Wierman[4] proved that, on a large subclass of bipartite graphs including the square lattice $\mathbb{Z}^2$, AB percolation does not occur for any value of p. On the other hand, Scheinerman and Wierman[5] proved that AB percolation occurs on some periodic two-dimensional graph. Wierman and Appel[6] proved that AB percolation occurs on the planar triangular lattice; note that this implies that AB percolation occurs on the close-packed graph $\mathbb{Z}^2_{\text{cp}}$ (see Fig. 1(c)). For further work on AB percolation, see ref. 7 and references therein.

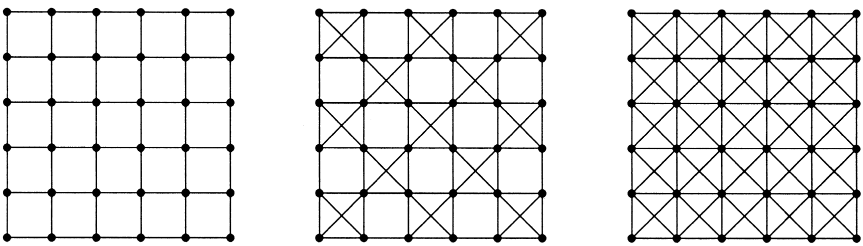
Let $\mathbb{Z}^2_{\text{hcp}}$ be the half close-packed graph of $\mathbb{Z}^2$, namely, the graph obtained from $\mathbb{Z}^2$ by adding the two diagonal edges into the faces of $\mathbb{Z}^2$ in a chessboard-like fashion (see Fig. 1(b)). Note that $\mathbb{Z}^2_{\text{hcp}}$ is the line graph (sometimes called covering graph) of $\mathbb{Z}^2$. One of the goals of this paper is to study the AB percolation model on the graph $\mathbb{Z}^2_{\text{hcp}}$. As mentioned in the previous paragraph, AB percolation occurs on $\mathbb{Z}^2_{\text{cp}}$ but does not occur on $\mathbb{Z}^2$, so it is natural to ask about what happens in the "intermediate case" of $\mathbb{Z}^2_{\text{hcp}}$.

Here we answer this question as follows:

Theorem 1.1. Let p_c be the critical probability for the site percolation on $\mathbb{Z}^2$. Then, AB percolation occurs on $\mathbb{Z}^2_{\text{hcp}}$ for all $p \in (1-\sqrt{1-p_c}, \sqrt{1-p_c})$.

The summary of 19 estimates provided by Ziff and Sapoval[8] suggests that $p_c \approx 0.5927$. Wierman[9] gave a rigorous upper bound $p_c \leqslant 0.679492$, which implies that $(0.434, 0.566) \subset (1-\sqrt{1-p_c}, \sqrt{1-p_c}) \neq \varnothing$.

Analogously to AB *site* percolation model which was discussed above, one can define AB *bond* percolation model on a graph G. Namely, for any $e \in E_G$, independently, we label it "A" with probability p, and label it "B" with probability $1-p$. As before, $X(e) \in \{A, B\}$ stands for the state of e.



(a): $\mathbb{Z}^2$ (b): $\mathbb{Z}^2_{hcp}$ (c): $\mathbb{Z}^2_{cp}$

Fig. 1. The graphs $\mathbb{Z}^2$, $\mathbb{Z}^2_{\text{hcp}}$ and $\mathbb{Z}^2_{\text{cp}}$.

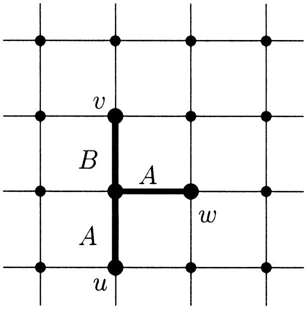


Fig. 2. u is connected to v, v is connected to w, but u is not connected to w by an AB bond path.

We call a self-avoiding path $\pi = \{v_0, e_0, v_1, e_1,..., e_{n-1}, v_n\}$ on G an AB bond path if $X(e_i) \neq X(e_{i+1})$ for all $0 \leqslant i < n-1$; and we say that AB bond percolation occurs if there exists an infinite AB bond path on G.

To the best of our knowledge, the AB bond percolation model was not yet considered in the literature. At first sight it may seem that the reason of that is the usual idea of reducing bond percolation to site percolation. However, for the case of AB percolation, it turns out that this idea does not work. The explanation for this is the fact that, unlike the case of AB site model, AB bond percolation is not transitive, i.e., if u is connected to v and v is connected to w by an AB bond path, this does not imply that u is connected to w by an AB bond path (see Fig. 2). This of course makes the problem more difficult, but anyway the question about whether there exists an infinite AB bond path is relevant. The following theorem gives a partial answer to this question (and shows, together with Theorem 1.1, that AB bond percolation on the two-dimensional square lattice is indeed substantially different from AB site percolation on its line graph).

Theorem 1.2. AB bond percolation does not occur on $\mathbb{Z}^2$ when $p = \frac{1}{2}$.

Remark. Taking into account the result of Appel,[10] it is reasonable to conjecture that the percolation probability function for AB bond model on $\mathbb{Z}^2$ is non-decreasing on interval $[0, \frac{1}{2}]$. Combined with the symmetry of the model and Theorem 1.2, this suggests that AB bond percolation does not occur for any value of p.

2. PROOFS

Define a random field Y on the vertex set of $\mathbb{Z}^2$ as follows. For each edge of the graph $\mathbb{Z}^2$, we insert a new vertex into its center. For any inserted vertex w, independently, we label it "A" (write $X(w)=A$) with probability p, and label it "B" (write $X(w)=B$) with probability $1-p$. For any fixed $v=(i,j)\in\mathbb{Z}^2$, let $w_1=(i-\frac{1}{2},j)$, $w_2=(i,j-\frac{1}{2})$, $w_3=(i+\frac{1}{2},j)$, $w_4=(i,j+\frac{1}{2})$ be the four inserted vertices nearest to v. Define the random variable

$$Y(v)=1-\mathbf{1}_{\{X(w_1)=X(w_2)=X(w_3)=X(w_4)\}}.$$

Clearly, the random field $Y=\{Y(v)\colon v\in\mathbb{Z}^2\}$ is dependent; denote the law of Y by μ_p.

Now, let μ and ν be two Borel probability measures on $\Omega:=\{0,1\}^{\mathbb{Z}^2}$. As usual, we say that μ dominates ν, if for any continuous increasing function f it holds that

$$\int f\,d\mu\geqslant\int f\,d\nu.$$

Denote by P_α the product measure with parameter $\alpha\in[0,1]$ on Ω. We have

Lemma 2.1. For any fixed $p\in[0,1]$, μ_p dominates $P_{1-q(p)}$, where $q(p)=\max\{p^2,(1-p)^2\}$.

Proof. We prove this lemma in two steps. ▮

Step 1. Let $B(n)=[-n,n]^2\cap\mathbb{Z}^2$, $n\in\mathbb{N}$, and let $\partial B(n)=B(n)\setminus B(n-1)$ be the boundary of $B(n)$. Let us enumerate $\mathbb{Z}^2=\{v_1,v_2,\ldots\}$ as follows: first, let v_1 be the origin, and then we order the vertices in $\partial B(1)$, $\partial B(2)$,... in turn. While ordering the vertices in $\partial B(n)$, we begin at vertex $(n,n-1)$ and put $v_{\sigma(n)+1}=(n,n-1)$, where $\sigma(n)=|B(n-1)|$. Then, along the clockwise direction, order the other vertices in $\partial B(n)$ in turn (in a way that $v_{\sigma(n+1)}=(n,n)$), and so on. So, a special ordering of $\mathbb{Z}^2$ was constructed (see Fig. 3).

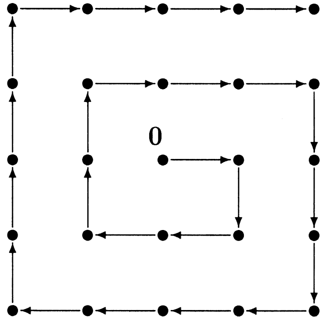


Fig. 3. At step 1 of the proof of Lemma 2.1, we order $\mathbb{Z}^2$ in this way.

Step 2. It follows from well-known standard arguments (see e.g., Lemma 1.1 of ref. 11 and references there), that it is sufficient to prove the following: for the special ordering defined on step 1, for any $n_1 < n_2 < \cdots < n_j < n_{j+1}$ and any choice of $\gamma_1, \gamma_2,..., \gamma_j \in \{0, 1\}$, it holds that

$$\mu_p(Y(v_{n_{j+1}}) = 1 \mid Y(v_{n_1}) = \gamma_1,..., Y(v_{n_j}) = \gamma_j) \geqslant 1 - \max\{p^2, (1-p)^2\} \tag{1}$$

whenever $\mu_p(Y(v_{n_1}) = \gamma_1,..., Y(v_{n_j}) = \gamma_j) > 0$.

If all the neighbors of $v_{n_{j+1}}$ are not in $\{v_{n_i}: 1 \leqslant i \leqslant j\}$, then, by definition of the random field Y, $Y(v_{n_{j+1}})$ is independent of $\{Y(v_{n_i}): 1 \leqslant i \leqslant j\}$. So, in this case the left-hand side of (1) equals $1-(p^4+(1-p)^4)$, which is greater than or equal to $1-q(p)$.

If $v_{n_{j+1}}$ is a neighbor of some vertex v_{n_i}, $1 \leqslant i \leqslant j$, let E be the set of edges which connect $v_{n_{j+1}}$ to its neighbors in $\{v_{n_i}: 1 \leqslant i \leqslant j\}$. Let V be set of inserted vertices which lie in the centers of edges of E. By the construction on step 1, it is clear that, for all given $n_1 < n_2 < \cdots < n_j < n_{j+1}$, we have $1 \leqslant |V| = |E| \leqslant 2$ (in the case when $v_{n_{j+1}}$ has at least one neighbor among $\{v_{n_j}: 1 \leqslant i \leqslant j\}$).

Now, by definition of the random field $Y = \{Y(v): v \in \mathbb{Z}^2\}$, $Y(v_{n_{j+1}})$ is independent of $\{Y(v_{n_i}): 1 \leqslant i \leqslant j\}$ conditioned on $\{X(w): w \in V\}$. Then, it is elementary to get that, for the case $|V| = 1$, the left-hand side of (1) is bounded from below by $1 - \max\{p^3, (1-p)^3\}$ (which is greater than or equal to $1-q(p)$), while for the case $|V| = 2$, the left-hand side of (1) is bounded from below by $1-q(p)$. This concludes the proof of Lemma 2.1. ∎

Now we are able to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us think about AB percolation on $\mathbb{Z}^2_{\text{hcp}}$ as AB percolation on the line graph of $\mathbb{Z}^2$ (two inserted vertices are adjacent iff the corresponding edges of $\mathbb{Z}^2$ have common vertex). Note that if $p \in (1-\sqrt{1-p_c}, \sqrt{1-p_c})$, then $1-q(p) > p_c$, so on $\mathbb{Z}^2$ there exists an infinite cluster of sites with Y-value 1. Now, if $Y(v) = 1$, the inserted vertices around v all belong to the same AB cluster. Hence, the infinite cluster of 1-s on $\mathbb{Z}^2$ induces an infinite AB cluster on its line graph, and so the proof of Theorem 1.1 is finished. ∎

To prove Theorem 1.2, we consider the following randomly oriented lattice. For any vertex $v = (i, j) \in \mathbb{Z}^2$, we call v *even* if $i+j$ is even and call v *odd* otherwise. For any edge e of $\mathbb{Z}^2$, let $v_{\text{even}}(e)$, $v_{\text{odd}}(e)$ be its even and odd endpoints respectively. Define a randomly oriented bond percolation model on $\mathbb{Z}^2$ as follows: For all $e \in \mathbb{Z}^2$, independently, let e have the orientation from $v_{\text{even}}(e)$ to $v_{\text{odd}}(e)$ with probability p and let e have the opposite orientation with probability $1-p$.

Now, consider the AB bond percolation model on $\mathbb{Z}^2$. Let e_0 be the edge connecting the sites (0, 0) and (0, 1), and suppose, without loss of generality, that its label is "*A*." Let us try to construct an infinite AB bond path beginning in (0, 0) and whose first bond is e_0. It is straightforward to see that such a path will always need "*A*" when passing from an even vertex to an odd one, and "*B*" otherwise. This argument shows that the following proposition holds.

Proposition 2.2. AB bond percolation occurs in $\mathbb{Z}^2$ if and only if there exists an infinite oriented path on the randomly oriented lattice described above.

Remark. Proposition 2.2 can be generalized to any bipartite graph G. In other words, for any bipartite graph G, one can study percolation on a randomly oriented graph defined as above instead of studying AB bond percolation model.

Proof of Theorem 1.2. Grimmett[12] considered another randomly oriented lattice: Each horizontal edge is oriented rightwards with probability p, and leftwards otherwise. Each vertical edge is oriented upwards with probability p, and downwards otherwise. It is straightforward to see that, when $p=\frac{1}{2}$, the two randomly oriented lattices are the same. As noted in refs. 7 and 12, in this case they are equivalent to the general bond percolation model on $\mathbb{Z}^2$. Then, by Proposition 2.2 and the fact that bond percolation in $\mathbb{Z}^2$ does not occur for $p=\frac{1}{2}$, Theorem 1.2 follows immediately. ∎

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